JOURNAL OF APPROXIMATION THEORY 83, 330 341 (1995)

Positive Bernstein–Sheffer Operators

PAUL SABLONNIERE

Laboratoire LANS, INSA de Rennes, Rennes 35043, France

Communicated by Manfred v. Golitschek

Received February 3, 1994; accepted in revised form September 22, 1994

Let $h(t)=\sum_{n\geqslant 1} h_n t^n$, $h_1>0$, and $\exp(xh(t))=\sum_{n\geqslant 0} p_n(x)\, t^n/n!$. For $f\in C[0,1]$, the associated Bernstein–Sheffer operator of degree n is defined by $B_n^h f(x)=p_n^{-1}\sum_{k=0}^{n} f(k/n)\binom{n}{k}\, p_k(x)\, p_{n-k}(1-x)$ where $p_n=p_n(1)$. We characterize functions h for which B_n^h is a positive operator for all $n\geqslant 0$. Then we give a necessary and sufficient condition insuring the uniform convergence of $B_n^h f$ to f. When h is a polynomial, we give an upper bound for the error $\|f-B_n^h f\|_\infty$. We also discuss the behavior of $B_n^h f$ when h is a series with a finite or infinite radius of convergence.
© 1995 Academic Press, Inc.

1. INTRODUCTION AND DEFINITIONS

A sequence of polynomials $s_n(x)\in\mathbb{P}_n$ is called a *Sheffer sequence* [6] if it is generated by an expansion of type

$$g(t)\exp(xh(t))=\sum_{n\geqslant 0} s_n(x)\,\frac{t^n}{n!} \tag{1.1}$$

where

$$g(t)=\sum_{n\geqslant 0} g_n t^n \qquad (g_0\neq 0) \tag{1.2}$$

and

$$h(t)=\sum_{n\geqslant 1} h_n t^n \qquad (h_1\neq 0). \tag{1.3}$$

With this Sheffer sequence is associated the sequence of polynomials $p_n(x)\in\mathbb{P}_n$ of binomial type generated by

$$\exp(xh(t))=\sum_{n\geqslant 0} p_n(x)\,\frac{t^n}{n!}. \tag{1.4}$$

0021-9045/95 $12.00
Copyright © 1995 by Academic Press, Inc.
All rights of reproduction in any form reserved.

Thanks to the properties of the exponential function, it is easy to prove that

$$p_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}p_k(x)\,p_{n-k}(y) \tag{1.5}$$

and also

$$s_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}s_k(x)\,p_{n-k}(y). \tag{1.6}$$

We define the *Bernstein–Sheffer operator* of degree n, associated with the function h (or with the corresponding sequence $p_n(x)$) by

$$B_n^h f(x)=\frac{1}{p_n(1)}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}p_k(x)\,p_{n-k}(1-x) \tag{1.7}$$

for $f\in C[0,1]$, provided that $p_n(1)\neq 0$ for all $n\geqslant 0$. The classical Bernstein operator corresponds to $h(t)=t$ and $p_n(x)=x^n$ (see e.g., G. G. Lorentz [4]). In this paper, we prove the following theorems in which we use the notation $e_i(x)=x^i$, $i\geqslant 0$, for monomials. As shown in Section 2.1, it suffices to consider the case $h_1>0$.

THEOREM 1. *B_n^h is a positive operator on $C[0,1]$ for all $n\geqslant 0$ if and only if $h_n\geqslant 0$ for all $n\geqslant 2$.*

THEOREM 2. *Assume that h satisfies the conditions of Theorem 1, then*

(i) *B_n^h is an isomorphism of $\mathbb{P}_n$ preserving the degree, i.e., $B_n^h p\in\mathbb{P}_k$ whenever $p\in\mathbb{P}_k$, $0\leqslant k\leqslant n$.*

(ii) *In addition, one has $B_n^h e_0=e_0$, $B_n^h e_1=e_1$ and $B_n^h e_2=e_2+a_n(e_1-e_2)$, where $a_n=1/n+((n-1)/n)(r_{n-2}/p_n)$, $p_n=p_n(1)$, $r_n=r_n(1)$,* the sequence *$\{r_n(x)\}$ being generated by*

$$h''(t)\exp(xh(t))=\sum_{n\geqslant 0} r_n(x)\frac{t^n}{n!}.$$

THEOREM 3. *Assume that h satisfies the conditions of Theorem 1, then*

(i) *$B_n^h f$ converges uniformly to $f\in C[0,1]$, when n tends to infinity, if and only if the condition $\lim_{n\to+\infty}(r_{n-2}/p_n)=0$ holds.*

(ii) *More specifically, when* $(r_{n-2}/p_n) = 0\,(1/n)$, *then there exists an integer* $k \geqslant 1$ *for which* $h \in \mathbb{P}_k$ *and we have*:

$$\|f - B_n^h f\|_\infty \leqslant \left(1 + \frac{1}{2}\sqrt{k}\right)\omega\left(f, \frac{1}{\sqrt{n}}\right)$$

where ω *is the modulus of continuity of* f.

This last result shows that the classical Bernstein operators ($k = 1$) could be considered as the best positive Bernstein–Sheffer operators associated with functions h which are polynomials. It would be nice *to characterize the class of functions* h, *satisfying the conditions of Theorem* 1, *for which condition* (i) *of Theorem* 3 *is also satisfied, i.e., for which* $\lim B_n^h f = f$ *for all* $f \in C[0, 1]$.

As shown in examples, this class is not reduced to polynomials and there exist various generating functions of arbitrary convergence radii, for which condition (i) is satisfied or not. We hope that this question is of interest for the reader and we leave it open at this moment.

Here is an outline of the paper: in Section 2, we prove Theorems 1 and 2. In Section 3, we prove Theorem 3. Finally, in Section 4, we give examples of series h satisfying or not condition (i) of Theorem 3.

2. Characterization and Properties of Positive Bernstein–Sheffer Operators

2.1. *Proof of Theorem* 1. First, we observe that if we set $\bar{h}(t) = h(-t) = -h_1 t + h_2 t^2 + h_3 t^3 + \cdots$ then the operators B_n^h and $B_n^{\bar{h}}$ coincide. For, $\exp(xh(-t)) = \sum_{n \geqslant 0} p_n(x)(-t)^n/n! = \sum_{n \geqslant 0} \bar{p}_n(x)\, t^n/n!$ implies $\bar{p}_n(x) = (-1)^n p_n(x)$. Replacing in (1.7) gives immediately the desired result.

(1) The proof of the necessary condition is made by induction on the degree. First, $B_1^h f$ exists iff $h_1 \neq 0$ since $p_1(x) = h_1 x$. Now, consider the expression

$$B_2^h f(x) = \frac{1}{p_2(1)}\{f_0 p_2(1-x) + 2f_1 p_1(x) p_1(1-x) + f_2 p_2(x)\}$$

where $f_i = f(i/2)$ for $i = 0, 1, 2$. Taking $f_0 = f_2 = 0$, $f_1 = 1$, we obtain

$$B_2^h f(x) = 2p_1(x)\, p_1(1-x)/p_2(1) \geqslant 0.$$

This implies $p_2(1) > 0$. Now, taking $f_0 = f_1 = 0$ and $f_2 = 1$, we obtain $p_2(x)/p_2(1) \geqslant 0$, therefore $p_2(x) \geqslant 0$. As $p_2(0) = 0$, we must have $p_2'(0) = 2!h_2 \geqslant 0$.

Assume that the following property is true

(P) $h_1 \neq 0$, $h_1 h_{2i+1} \geqslant 0$ for $1 \leqslant i \leqslant k-1$ and $h_{2i} \geqslant 0$ for $1 \leqslant i \leqslant k$,

and let us prove that $h_1 h_{2k+1} \geqslant 0$ and $h_{2k+2} \geqslant 0$.

$$B^h_{2k+1} f(x) = \frac{1}{p_{2k+1}(1)} \sum_{i=0}^{2k+1} \binom{2k+1}{i} f_i p_i(x) p_{2k+1-i}(1-x)$$

where $f_i = f(i/(2k+1))$.

Taking $f_0 = \cdots = f_{2k} = 0$ and $f_{2k+1} = 1$, we obtain

$$p_{2k+1}(x)/p_{2k+1}(1) \geqslant 0.$$

Taking $f_0 = \cdots = f_{2k-1} = f_{2k+1} = 0$ and $f_{2k=1}$, we obtain

$$p_{2k}(x)\, p_1(1-x)/p_{2k+1}(1) \geqslant 0.$$

Since, by induction, $p_{2k}(x) \geqslant 0$ because B^h_{2k} is a positive operator, we see that $p_{2k+1}(1)$ has the sign of p_1 (i.e., of h_1) and consequently by the first inequality, $p_{2k+1}(x)$ has also the sign of h_1 on $[0, 1]$. As $p_{2k+1}(0) = 0$, we have $p'_{2k+1}(0) = (2k+1)!\, h_{2k+1}$ which implies $h_1 h_{2k+1} \geqslant 0$, q.e.d. In a similar way, one deduces that $h_{2k+2} \geqslant 0$ from the positivity of the operator B^h_{2k+2}. From the invariance of the operators by changing $h(t)$ into $h(-t)$, we can restrict property (P) to $h_1 > 0$ and $h_i \geqslant 0$ for all $i \geqslant 2$.

(2) The condition of Theorem 1 is also sufficient since

$$p_n(x) = \sum_{k=1}^{n} x^k \beta_{nk}(h_1, 2h_2, ..., (n-k+1)!\, h_{n-k+1}),$$

where the β_{nk} are the exponential Bell polynomials (see e.g. Riordan [5], chapters 4, 5), therefore $p_n(x) \geqslant 0$ for $x \geqslant 0$ and B^h_n is a positive operator. This can be deduced also from:

$$\frac{d^k}{dx^k} \exp(xh(t)) = [h(t)]^k \exp(xh(t)) = \sum_{n \geqslant k} \frac{d^k}{dx^k} p_n(x)\, t^n/n!$$

For $x = 0$, we obtain $[h(t)]^k = \sum_{n \geqslant k} d^k/dx^k\, p_n(0)\, t^n/n!$ Since h has positive coefficients, the derivatives of p_n at $x = 0$ are positive, hence $p_n(x) \geqslant 0$ for $x \geqslant 0$, and finally $B^h_n f(x) \geqslant 0$ if $f(x) \geqslant 0$ for $x \in [0, 1]$. ∎

In the rest of the paper, we assume that h satisfies the conditions of Theorem 1.

2.2. *Proof of Theorem* 2. $B_n^h e_0(x) = 1/p_n \sum_{k=0}^n \binom{n}{k} p_k(x) p_{n-k}(1-x)$ is equal to $e_0(x)=1$, from (1.5) with $y=1-x$. Now, with $g(t)=h'(t)$, we have:

$$g(t)\exp(xh(t)) = \frac{1}{x}\frac{d}{dt}\exp(xh(t)) = \sum_{n\geq 0} \frac{p_{n+1}(x)}{x} t^n/n!.$$

Therefore $\{s_n(x)=p_{n+1}(x)/x\}$ is the Sheffer sequence associated with g and we deduce from (1.6)

$$s_{n-1}(1)=p_n(1)=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{p_{k+1}(x)}{x} p_{n-k-1}(1-x)$$

or

$$xp_n(1)=\sum_{k=1}^{n}\binom{n-1}{k-1} p_k(x)\, p_{n-k}(1-x) \tag{2.1}$$

or

$$xp_n(1)=\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k} p_k(x)\, p_{n-k}(1-x)$$

which is equivalent to $B_n^h e_1 = e_1$.

Let us define the Sheffer sequences $\{q_n(x)\}$ and $\{r_n(x)\}$ associated with $g_1(t)=(h'(t))^2$ and $g_2(t)=h''(t)$ (we assume that $h''(0)=2h_2>0$). Then we have

$$q_n(1)=\sum_{k=0}^{n}\binom{n}{k} q_k(x)\, p_{n-k}(1-x)$$

$$r_n(1)=\sum_{k=0}^{n}\binom{n}{k} r_k(x)\, p_{n-k}(1-x). \tag{2.2}$$

On the other hand, we obtain

$$\frac{d^2}{dt^2}\exp(xh(t)) = (xh''(t)+x^2(h'(t))\exp(xh(t)) = \sum_{n\geq 0} p_{n+2}(x)\, t^n/n!. \tag{2.3}$$

From (2.2) and (2.3), we deduce successively

$$p_{n+2}(x) = xr_n(x) + x^2 q_n(x) \tag{2.4}$$

$$\begin{aligned} xr_n(1) + x^2 q_n(1) &= \sum_{k=0}^{n} \binom{n}{k} p_{k+2}(x)\, p_{n-k}(1-x) \\ &= \sum_{k=0}^{n} \frac{k(k-1)}{(n+1)(n+2)} \binom{n+2}{k} p_k(x)\, p_{n+2-k}(1-x) \\ &= \sum_{k=2}^{n} \frac{k(k-1)}{n(n-1)} \binom{n}{k} p_k(x)\, p_{n-k}(1-x). \end{aligned}$$

Using (2.1), we obtain:

$$e_2(x) = x^2 = \frac{1}{q_{n-2}(1)} \sum_{k=1} \left\{\frac{k(k-1)}{n(n-1)} - \frac{k}{n}\,\frac{r_{n-2}(1)}{p_n(1)}\right\} \binom{n}{k} p_k(x)\, p_{n-k}(1-x). \tag{2.5}$$

For sake of simplicity, denote $p_n = p_n(1)$, $q_n = q_n(1)$, $r_n = r_n(1)$, and $b_{nk}(x) = \binom{n}{k} p_k(x)\, p_{n-k}(1-x)$. With these notations, we obtain:

$$B_n^h e_2(x) - e_2(x) = \frac{1}{p_n} \sum_{k=1}^{n} \left\{\frac{k}{n} - \frac{(k-1)\, p_n}{(n-1)\, q_{n-2}} + \frac{r_{n-2}}{q_{n-2}}\right\} \frac{b}{k}\, b_{nk}(x). \tag{2.6}$$

From (2.4) with $x = 1$, (2.1), (2.5) and (2.6), we deduce respectively:

$$e_1(x) - e_2(x) = \frac{1}{q_{n-2}} \sum_{k=1}^{n} \frac{k(n-k)}{n(n-1)}\, b_{nk}(x).$$

Comparing these two identities gives

$$B_n^h e_2 - e_2 = \frac{q_{n-2}}{p_n} \left(\frac{1}{n} + \frac{r_{n-2}}{q_{n-2}}\right) (e_1 - e_2) = a_n (e_1 - e_2)$$

where

$$a_n = \frac{1}{n} \left(\frac{q_{n-2}}{p_n} + \frac{n r_{n-2}}{p_n}\right) = \frac{1}{n} \left(\frac{q_{n-2} + n r_{n-2}}{q_{n-2} + r_{n-2}}\right)$$

or

$$a_n = \frac{1}{n} \left(1 + (n-1)\,\frac{r_{n-2}}{p_n}\right), \qquad \text{therefore} \quad B_n^h e_2 = e_2 + a_n (e_1 - e_2) \in \mathbb{P}_2.$$

More generally, we have:

$$\frac{d^k}{dt^k}(\exp(xh(t))) = \sum_{n \geqslant 0} p_{n+k}(x)\, t^n/n! \tag{2.7}$$

$$= (xg_1(t) + x^2 g_2(t) + \cdots + x^k g_k(t)) \exp(xh(t)) \tag{2.8}$$

where $g_i(t)$ is a product of derivatives of $h(t)$.

$$\text{Expanding } g_i(t) \exp(xh(t)) = \sum_{n \geqslant 0} \alpha_{in}(x)\, t^n/n! \tag{2.9}$$

for $i = 1, ..., k$, we obtain k Sheffer sequences, therefore, by (1.6), (2.7), (2.8) and (2.9), we get

$$x\alpha_{1n}(1) + x^2\alpha_{2n}(1) + \cdots + x^h\alpha_{kn}(1)$$

$$= \sum_{i=0}^{n} \binom{n}{i} p_{i+k}(x)\, p_{n-i}(1-x)$$

$$= \sum_{i=k}^{n-k} \frac{i(i-1)\cdots(i-k+1)}{n(n-1)\cdots(n-k+1)} \binom{n+k}{i} p_i(x)\, p_{n+k-i}(1-x)$$

$$= \frac{(n+k)^k\, p_{n+k}}{n(n+1)\cdots(n+k-1)} \left\{ \frac{1}{p_{n+k}} \sum_{i=k}^{n+k} \frac{i(i-1)\cdots(i-k+1)}{(n+k)^k} b_{n+k,i}(x) \right\}. \tag{2.10}$$

But since $i(i-1)\cdots(i-k+1) \in \mathbb{P}_k[i]$, the sum between brackets is a linear combination of the $B_{n+k}(e_j) = 1/p_{n+k} \sum_{i=0}^{n+k} (i/(n+k))^j\, b_{n+k,i}(x)$ for $1 \leqslant j \leqslant k$. If we assume that $B_{n+k}e_j \in \mathbb{P}_j$ for $0 \leqslant j \leqslant k-1$, the above expression shows that $B_{n+k}e_k \in \mathbb{P}_k$ for all $n \geqslant 0$, i.e., B_n^h preserves the degree of polynomials, q.e.d. ∎

3. Convergence of $B_n^h f$ to $f \in C[0, 1]$

In this section, we assume that h satisfies the conditions of Theorem 1, i.e. B_n^h is a positive operator on $C[0, 1]$, for all $n \geqslant 0$. We want to prove Theorem 3. For this, we need the following

Lemma. *Let h be an entire function, then the sequence $\{(n+1)r_n/p_{n+2}\}$ is bounded by $k-1$ if and only if $h \in \mathbb{P}_k$.*

Proof. (1) Assume that $h(t) = h_1 t + \cdots + h_k t^k \in \mathbb{P}_k$ with h_1 and $h_k > 0$. Then we compute:

$$th''(t) \exp(h(t)) = (2h_2 t + 6h_3 t^2 + \cdots + k(k-1) h_k t^{k-1}) \sum_{n \geq 0} p_n t^n/n!$$

$$= \sum_{n \geq 1} \left(\frac{2h_2 p_{n-1}}{(n-1)!} + \cdots + \frac{k(k-1) h_k p_{n-k+1}}{(n-k+1)!} \right) t^n$$

(with the convention $p_j = 0$ for $j < 0$).

$$\frac{d}{dt} (th''(t) \exp(h(t))) = \sum_{n \geq 0} \left(\frac{2h_2 p_n}{n!} + \cdots + \frac{k(k-1) h_k p_{n-k+2}}{(n-k+2)!} \right) (n+1) t^n \tag{3.1}$$

$$h'(t) \exp(h(t)) = \sum_{n \geq 0} \left(\frac{h_1 p_n}{n!} + \frac{2h_2 p_{n-1}}{(n-1)!} + \cdots + \frac{kh_k p_{n-k+1}}{(n-k+1)!} \right) t^n$$

$$\frac{d}{dt} h'(t) \exp(h(t)) = \sum_{n \geq 0} \left(\frac{h_1 p_{n+1}}{(n+1)!} + \frac{2h_2 p_n}{n!} + \cdots + \frac{kh_k p_{n-k+2}}{(n-k+2)!} \right) (n+1) t^n. \tag{3.2}$$

Comparing the brackets in (3.1) and (3.2) we obtain, for $k \geq 2$ and $t \geq 0$:

$$\frac{d}{dt} (th''(t) e^{h(t)}) \leq (k-1) \frac{d}{dt} (h'(t) e^{h(t)}). \tag{3.3}$$

But (3.1) is also equal to $\sum_{n \geq 0} (n+1) r_n t^n/n!$. Similarly, (3.2) is equal to $\sum_{n \geq 0} p_{n+2} t^n/n!$ and (3.3) gives for all $n \geq 0$

$$(n+1) r_n \leq (k-1) p_{n+2} \tag{3.4}$$

whence the result on the sequence $\{(n+1) r_n/p_{n+2}\}$.

(2) Reciprocally, if (3.4) is true, then (3.3) is true for all $t \geq 0$. A first integration gives:

$$th''(t) e^{h(t)} \leq (k-1)(h'(t) e^{h(t)} - h_1)$$
$$\leq (k-1) h'(t) e^{h(t)}$$

whence

$$th''(t) \leq (k-1) h'(t)$$

or

$$th''(t) + h'(t) \leq kh'(t).$$

A second integration gives $th'(t) \leqslant kh(t)$ and a third one $0 \leqslant h(t) \leqslant h(1)t^k$ for all $t \geqslant 0$. By Rouché's theorem, this implies that $h \in \mathbb{P}_k$.

Proof of Theorem 3. (i) is an immediate consequence of Korovkin's theorem (see [3]): Since B_n^h is a positive operator and $B_n^h e_i = e_i$ for $i = 0, 1$, it suffices to have $\lim B_n^h e_2 = e_2$ which is equivalent, in view of Theorem 2(ii) to condition (i) of Theorem 3.

(ii) $r_{n-2}/p_n = O(1/n)$ means that the sequence nr_{n-2}/p_n is bounded, so from the lemma above, $h \in \mathbb{P}_k$ for some $k \geqslant 1$. Now, for all $\delta > 0$ and $f \in C[0, 1]$, we have:

$$|f(t) - f(x)| \leqslant \left(1 + \frac{|t-x|}{\delta}\right) \omega(f, \delta).$$

Let $\varphi(t) = f(t) - f(x)$ and $\psi(t) = |t - x|$: since B_n^h is a positive operator and $B_n^h e_0 = e_0$, $|B_n^h \varphi(x)| \leqslant (1 + (1/\delta) B_n^h \psi(x))\, \omega(f, \delta)$.

By Schwarz's inequality:

$$(B_n^h \psi)^2 \leqslant (B_n^h e_0)(B_n^h \psi^2) = B_n^h \psi^2$$

$$B_n^h \psi^2(x) = B_n^h(e_2 - 2xe_1 + x^2) = a_n x(1-x) \leqslant \tfrac{1}{4} a_n$$

from which we deduce, for $x \in [0, 1]$:

$$|B_n^h f(x) - f(x)| \leqslant \left(1 + \frac{\sqrt{a_n}}{2\delta}\right) \omega(f, \delta).$$

Taking $\delta = 1/\sqrt{n}$ and using the inequality $na_n \leqslant k$ (since $h \in \mathbb{P}_k$), we obtain the desired result. ∎

4. Examples of Generating Functions h

4.1. *Functions h for Which $B_n^h f$ Does Not Converge to f.* For any $a > 0$, consider the series

$$h(t) = -ln\left(1 - \frac{t}{a}\right) = \sum_{m \geqslant 1} \frac{t^m}{ma^m}.$$

Its radius of convergence is a. We compute successively:

$$\exp(h(t)) = \frac{a}{a-t} = \sum_{n \geqslant 0} p_n \frac{t^n}{n!}, \qquad \text{with} \quad p_n = \frac{n!}{a^n}, \; h''(t) = \frac{1}{(a-t)^2}$$

and

$$h''(t)\exp(h(t))=\frac{a}{(a-t)^3}=\sum_{n\geq 0} r_n\frac{t^n}{n!},\qquad \text{with}\quad r_n=\frac{1}{2}\frac{(n+2)!}{a^{n+2}}.$$

Therefore

$$\lim_{n\to\infty}\frac{r_n}{p_{n+2}}=\frac{1}{2}\neq 0.$$

This shown that there exist functions h, with an arbitrary large radius of convergence, which do not satisfy condition (i) of Theorem 3, i.e. for which $B_n^h f$ does not converge to f when n tends to infinity. Here, for example, $B_n^h e_2$ converges to $\frac{1}{2}(e_1+e_2)$.

4.2. *Functions h for Which $B_n^h f$ Converge to f.* (a) *h has a finite radius of convergence.* Consider

$$h(t)=\frac{t}{1-t}=\sum_{m\geq 1} t^m,$$

then it is known (see e.g. Riordan [5], p. 194 or Comtet [1], vol. 1, p. 165) that

$$\exp(h(t))=1+\sum_{n\geq 1}\left(\sum_{k=1}^{n}\frac{n!}{k!}\binom{n-1}{k-1}\right)\frac{t^n}{n!}.$$

Since,

$$h''(t)=\frac{2}{(1-t)^3},$$

we also get

$$h''(t)\exp(h(t))=\frac{2}{(1-t)^3}\exp\left(\frac{t}{1-t}\right)=\sum_{n\geq 0} 2L_n^{(2)}(-1)t^n$$

where

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{k+\alpha}\frac{(-x)^k}{k!}$$

are the generalized Laguerre polynomials. Finally, we obtain

$$\begin{cases} p_n=\displaystyle\sum_{k=1}^{n}\frac{n!}{k!}\binom{n-1}{k-1}=-(n-1)!\,\frac{d}{dx}L_n^{(0)}(-1)=(n-1)!\,L_{n-1}^{(1)}(-1)\\ r_n=2n!\,L_n^{(2)}(-1)\end{cases}$$

and

$$\frac{1}{2}\frac{r_n}{p_{n+2}} = \frac{1}{n+1}\frac{L_n^{(2)}(-1)}{L_{n+1}^{(1)}(-1)}.$$

By using recurrence relationships between Laguerre polynomials, it is possible to prove that this quantity behaves like $1/\sqrt{n+1}$ (Paszkowski [7], 1990, see also Szegö [8], Chapter XII), so we get

$$\lim_{n \to +\infty} \frac{r_n}{p_{n+2}} = 0.$$

This is an example of series $h(t)$, with a finite radius of convergence $R = 1$, satisfying condition (i) of Theorem 3.

(b) *h is an entire function.* Consider

$$h(t) = e^t - 1 = \sum_{m \geq 1} \frac{t^m}{m!},$$

then

$$\exp(h(t)) = \sum_{n \geq 0} \tilde{\omega}(n) \frac{t^n}{n!}, \qquad \text{where} \quad \tilde{\omega}(n) = \sum_{k=1}^{n} s(n, k)$$

is the sum of the Stirling numbers of the second kind (see e.g. Comtet [1], vol. 2, p. 45),

$$h''(t) \exp(h(t)) = \sum_{n \geq 0} \left(\sum_{j=0}^{n} \binom{n}{j} \tilde{\omega}(j) \right) \frac{t^n}{n!}.$$

Since

$$\sum_{j=0}^{n} \binom{n}{j} \tilde{\omega}(j) = \tilde{\omega}(n+1),$$

we get

$$\frac{r_n}{p_{n+2}} = \frac{\tilde{\omega}(n+1)}{\tilde{\omega}(n+2)}.$$

In Comtet ([1], p. 144, exercise 23), we find:

$$\tilde{\omega}(n) \sim (x_n + 1)^{-1/2} \exp\{n(x_n + x_n^{-1} - 1) - 1\}$$

where x_n is the unique root of $xe^x = n$. Then, in De Bruijn ([2], p. 25), we find

$$x_n \sim \log n \qquad \text{when} \quad n \to +\infty,$$

therefore, we obtain $\tilde{\omega}(n) \sim n^n/\sqrt{\log n}$ and

$$\frac{\tilde{\omega}(n)}{\tilde{\omega}(n+1)} \sim \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n$$

tends to zero when n tends to infinity.

This example shows that there exist entire functions h, which are not polynomials, for which B_n^h converges to f.

References

[1] L. Comtet, "Analyse Combinatoire," 2 vols., Presses Universitaires de France, Paris, 1970.

[2] N. G. De Bruijn, "Asymptotic Methods in Analysis," Dover, New York, 1981.

[3] P. P. Korovkin, "Linear Operators and Approximation Theory," Hindustan Publ., Dehli, 1960.

[4] G. G. Lorentz, "Bernstein Polynomials," University of Toronto, Toronto, 1953.

[5] J. Riordan, "Combinatorial Identities," Wiley, New York, 1968.

[6] I. M. Sheffer, Some properties of polynomial sets of type zero, *Duke Math. J.* **5** (1939), 590–622.

[7] S. Paszkowski, private communication, 1990.

[8] G. Szegö, "Orthogonal Polynomials," AMS Colloquium Publications, Vol. 23, Amer. Math. Soc. Providence, RI, 1975.